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Control-Theoretic Analysis of Low-Thrust Orbital Transfer Using Orbital Elements

Pini Gurfil*
Princeton University, Princeton, New Jersey 08544

Introduction

OVER the past few decades, there has been a significant progress in the research of advanced nonchemical propulsion systems for space missions. While providing high specific impulse, the new propulsion systems usually yield low thrust levels. Thus, to perform orbital transfer the propulsion system would be required to operate during most of the transfer phase or even all of it. The continuous-time thrusting necessitates a specialized design of low-thrust transfer trajectories.

The bulk of the previous low-thrust orbital transfer research has been devoted to the design of open-loop, optimal (minimum-energy, minimum-time) trajectories.[1,2] The transfer trajectory optimization procedures have utilized various dynamical models, formulated using inertial Cartesian coordinates[1,2] and classical or equinoctial osculating orbital elements.[3] Various numerical optimization methods, such as nonlinear collocation[3] and evolutionary programming,[4] have been extensively investigated.

Less attention has been given to the closed-loop orbital transfer problem, aimed at using state variable feedback in order to perform the orbital transfer in a more robust manner. Nevertheless, a few significant efforts in this regard have been recently reported.[5–7]

Issues of vast importance for current and future missions are the feasibility and performance of closed-loop low-thrust orbital transfers based on osculating orbital element feedback only. Within this context several methodologies have been proposed.[8,9] These works utilized Gauss' variational equations (GVEs) for designing nonlinear continuous-time orbital transfer feedback controllers. However, although Lyapunov stability for the closed-loop system has been shown, important aspects of the orbital transfer problem, such as controllability, were not considered, and the problem of transfers to parabolic orbits (escape trajectories) was not discussed. The purpose of this Note, therefore, is to derive a comprehensive and rigorous control-theoretic framework for both analysis and design of low-thrust orbital transfers using orbital elements feedback, by analyzing accessibility and stabilizability properties of the GVEs.

Background

We consider the class of control-affine systems Σ_A defined as follows:

$$\Sigma_A : \dot{\boldsymbol{x}} = \boldsymbol{g}_0(\boldsymbol{x}) + \sum_{i=1}^{m} u_i \boldsymbol{g}_i(\boldsymbol{x}) = \boldsymbol{g}_0(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u} \tag{1}$$

where $\boldsymbol{x} \in \mathcal{X}$, $\mathcal{X}$ is an open set on an n_x-dimensional differentiable manifold $\mathcal{M}$, $\boldsymbol{u} \in \mathbb{R}^m$, and $\boldsymbol{g}_i(\boldsymbol{x})$, $i = 0, 1, \ldots, m$ are smooth ($\mathcal{C}^\infty$) vector fields defined on $\mathcal{M}$. It is assumed that the control input $\boldsymbol{u}$ is admissible, namely, a measurable, bounded function $\boldsymbol{u} : [0, \infty) \to \mathbb{R}^m$. We say that an admissible input $\boldsymbol{u}$ is a feedback control law when $\exists K : \mathcal{X} \to \mathbb{R}^m$ such that $\boldsymbol{u} = K(\boldsymbol{x})$.

The first step in synthesizing a control input $\boldsymbol{u}$ for any dynamical system is to determine whether the system is controllable. To examine controllability properties of the nonlinear systems dwelt upon in the sequel, we shall adopt the notion of accessibility,[10] a weaker form of controllability, defined as follows.

Definition 1: Let $\mathcal{R}_T(\boldsymbol{x}_0)$ denote the set of states reachable from the initial state $\boldsymbol{x}_0$ in a finite time T using an admissible control $\boldsymbol{u}$. A system in class Σ_A is said to be accessible (from $\boldsymbol{x}_0$) if $\mathcal{R}_T(\boldsymbol{x}_0)$ has a nonempty interior in $\mathbb{R}^{n_x}$.

The notion of accessibility is weaker than controllability because the reachable set $\mathcal{R}_T(\boldsymbol{x}_0)$ is not required to be $\mathbb{R}^{n_x}$. To provide sufficient conditions for accessibility, we review a few basics of differential geometry. First, recall that the Lie bracket is a binary operation, which associates to an (ordered) pair of vector fields, $\boldsymbol{g}_0, \boldsymbol{g}_1$, the vector field

$$[\boldsymbol{g}_0, \boldsymbol{g}_1] = D\boldsymbol{g}_1 \cdot \boldsymbol{g}_0 - D\boldsymbol{g}_0 \cdot \boldsymbol{g}_1 \tag{2}$$

where $D\boldsymbol{g}_i$ denotes the Jacobian matrix of $\boldsymbol{g}_i$. The ad operator is iteratively defined by

$$ad^1_{g_0}\boldsymbol{g}_1 = [\boldsymbol{g}_0, \boldsymbol{g}_1], \qquad ad^{k+1}_{g_0}\boldsymbol{g}_1 = \left[\boldsymbol{g}_0, ad^k_{g_0}\boldsymbol{g}_1\right] \tag{3}$$

We can now write the smooth (differential geometric) distribution

$$\Delta(\boldsymbol{x}) = \operatorname{span}\left\{\boldsymbol{g}_0(\boldsymbol{x}), ad^{k+1}_{g_0}\boldsymbol{g}_i(\boldsymbol{x}), i = 1, \ldots m, k = 1, 2, \ldots\right\} \tag{4}$$

The following theorem provides a sufficient condition for accessibility.[10]

Theorem 1 accessibility rank condition: Any system in class Σ_A is accessible from $\boldsymbol{x}_0$ if $\dim \Delta(\boldsymbol{x}_0) = n$.

A concept related (but, in the nonlinear case, not necessarily identical) to accessibility is that of feedback stabilizability. The feedback stabilization problem is usually stated as follows: Given an $\boldsymbol{x}_d \in \mathcal{X}$, find a feedback control law $\boldsymbol{u} \in \mathbb{R}^m$ that renders $\boldsymbol{x}_d$ an asymptotically stable equilibrium (i.e., Lyapunov stable and attractive) of Σ_A. If such a (continuous) feedback exists, it is called a (continuous) internal asymptotic feedback stabilizer of Σ_A. A well-known necessary condition for the existence of a continuous local internal asymptotic feedback stabilizer (Brockett's test) is given by the following theorem.[10]

Theorem 2: Denote $\boldsymbol{y} = \boldsymbol{g}_0(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}$ and let $B_\varepsilon(\boldsymbol{x}, \boldsymbol{x}_d) \triangleq \{\boldsymbol{x} \in \mathcal{X} : \|\boldsymbol{x} - \boldsymbol{x}_d\| < \varepsilon\}$, $B_\varepsilon(\boldsymbol{u}, \boldsymbol{0}) \triangleq \{\boldsymbol{u} \in \mathbb{R}^m : \|\boldsymbol{u}\| < \varepsilon\}$, $B_\delta(\boldsymbol{y}, \boldsymbol{0}) \triangleq \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{y}\| < \delta\}$ with $\varepsilon, \delta > 0$ sufficiently small. If there exists a continuous asymptotic (local) internal feedback stabilizer for Σ_A, then the mapping $\boldsymbol{y} : \mathcal{X} \times \mathbb{R}^m \to \mathbb{R}^n$ is onto an open set containing the origin. □

Accessible systems might still fail to be stabilizeable by continuous-time feedback, a phenomenon known as the controllability-stabilizability gap.[10] We shall later see that in some cases the low-thrust orbital transfer problem constitutes an example for such systems.

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*Research Staff Member, Mechanical and Aerospace Engineering Department; pgurfil@princeton.edu. Senior Member AIAA.

Elliptic Orbital Transfer Problem

A commonly used set of coordinates for modeling the orbital state-space dynamics of a (possibly perturbed) motion in a central gravitational field are the classical osculating orbital elements, given by

$$\boldsymbol{\alpha} = [a, e, i, \Omega, \omega, M]^T \in \mathcal{O} \times \mathbb{S}^4 \tag{5}$$

where $\mathcal{O} \subset \mathbb{R}^2$ is an open set in $\mathbb{R}^2$; $\mathbb{S}^4$ is the 4-sphere; and $a, e, i, \Omega, \omega, M$ denote the semimajor axis, the eccentricity, the inclination, the longitude of the ascending node, the argument of the periapsis, and the mean anomaly along the orbit, respectively. Denoting the control inputs by $\boldsymbol{u}$, the Gauss variational equations, modeling the state-space dynamics, are given by

$$\Sigma_G(\boldsymbol{\alpha}, \boldsymbol{u}) : \dot{\boldsymbol{\alpha}} = \boldsymbol{g}_0(\boldsymbol{\alpha}) + G(\boldsymbol{\alpha})\boldsymbol{u} \tag{6}$$

where

$$\boldsymbol{g}_0(\boldsymbol{\alpha}) \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ n \end{bmatrix}$$

$$G(\boldsymbol{\alpha}) \triangleq \frac{1}{h} \begin{bmatrix} 2a^2 e s(\phi) & \dfrac{2a^2 p}{r} & 0 \\ p s(\phi) & (p+r)c(\phi) + re & 0 \\ 0 & 0 & rc(\phi+\omega) \\ 0 & 0 & \dfrac{rs(\phi+\omega)}{s(i)} \\ \dfrac{-pc(\phi)}{e} & \dfrac{(p+r)s(\phi)}{e} & \dfrac{-rs(\phi+\omega)c(i)}{s(i)} \\ \dfrac{b[pc(\phi)-2re]}{ae} & \dfrac{-b(p+r)s(\phi)}{ae} & 0 \end{bmatrix} \tag{7}$$

In Eq. (7) we have $c(\cdot) \triangleq \cos(\cdot)$ and $s(\cdot) \triangleq \sin(\cdot)$, $h = \sqrt{(\mu p)}$, $n = \sqrt{(\mu/a^3)}$, $b = a\sqrt{(1-e^2)}$, $p = a(1-e^2)$, $r = \|\boldsymbol{r}\| = p/(1 + e\cos\phi)$, and $\boldsymbol{u} = [u_r, u_\theta, u_h]^T \in \mathbb{R}^3$ comprises control forces in the radial, tangential, and normal directions, respectively. The variable ϕ appearing in Eq. (7) is the true anomaly, a function of M and e, explicitly given, for example, by the following series solution to Kepler's equation[11]:

$$\phi(M, e) = M + 2\sum_{l=1}^{\infty} \frac{1}{l}\left[\sum_{s=-\infty}^{\infty} J_s(-le)\left(\frac{1-\sqrt{1-e^2}}{e}\right)^{|l+s|}\right]\sin(lM) \tag{8}$$

where $J_s(\cdot)$ is a Bessel function of the first kind of order s. The union of all elliptic Keplerian orbits on $\Sigma_G(\alpha, \mathbf{0})$ is parameterized by

$$\Sigma_{\mathrm{GE}} = \{\boldsymbol{\alpha} : \alpha \in (R_M, \infty), e \in (0, 1), i \in [0, \pi], \{\Omega, \omega\} \in [0, 2\pi]\} \tag{9}$$

where R_M is the gravitational body (maximal) radius.

The elliptic continuous-feedback orbital transfer problem is to determine whether there exists a continuous internal asymptotic feedback stabilizer $\boldsymbol{u}(\boldsymbol{\alpha})$ that steers a spacecraft from an initial elliptic Keplerian orbit $\Sigma_0 \in \Sigma_{\mathrm{GE}}$ to a desired elliptic Keplerian orbit $\Sigma_d \in \Sigma_{\mathrm{GE}}$. The next sections discuss important aspects of the orbital transfer problem utilizing the concepts just presented.

Accessibility of the Gauss Variational Equations

Testing accessibility (or controllability) is an essential first step in designing control laws for any dynamical system because it indicates whether the system is globally or locally accessible, and, more importantly, it can be used as a stepping stone for the derivation of closed-loop controllers solving the elliptic continuous-feedback orbital transfer problem.

Our main result in this context is stated by the following lemma, which asserts that the dynamical system $\Sigma_G(\boldsymbol{\alpha}, \boldsymbol{u})$ is accessible from any initial set of orbital elements.

Lemma 1: System $\Sigma_G(\boldsymbol{\alpha}, \boldsymbol{u})$ is accessible from $\boldsymbol{\alpha}_0$, $\forall \boldsymbol{\alpha}_0 \in \mathcal{O} \times \mathbb{S}^4$.

Proof: Let $G(\boldsymbol{\alpha}) = [\boldsymbol{g}_1(\boldsymbol{\alpha}), \boldsymbol{g}_2(\boldsymbol{\alpha}), \boldsymbol{g}_3(\boldsymbol{\alpha})]$. One way of showing that the accessibility rank condition holds is to find six iterated Lie brackets that are linearly independent $\forall \boldsymbol{\alpha}_0 \in \mathcal{O} \times \mathbb{S}^4$. Consider the pairs $\Lambda = [[\boldsymbol{g}_0, \boldsymbol{g}_1], [\boldsymbol{g}_0, [\boldsymbol{g}_0, \boldsymbol{g}_1]], [\boldsymbol{g}_0, \boldsymbol{g}_2], [\boldsymbol{g}_0, [\boldsymbol{g}_0, \boldsymbol{g}_2]], [\boldsymbol{g}_0, \boldsymbol{g}_3], [\boldsymbol{g}_0, [\boldsymbol{g}_0, \boldsymbol{g}_3]]]$. The 6×6 matrix Λ is given in the Appendix. By inspecting the structure of Λ, it is evident that generally, for almost all $\boldsymbol{\alpha}_0 \in \mathcal{O} \times \mathbb{S}^4$, the matrix does not drop rank. To substantiate this observation, we have to examine when $\det\Lambda = 0$, or, equivalently, to find solutions to the following equation (obtained after some symbolic algebra manipulations on $\det\Lambda$):

$$\begin{aligned} \beta\left(e, \phi, \frac{\partial\phi}{\partial M}\right) = \mu^2\left(\frac{\partial\phi}{\partial M}\right)^8 \Bigg(& p\frac{\partial\phi}{\partial M}\{[-54c(\phi) + 18c(3\phi)]e^6 \\ &+ [-18c(2\phi) + 6]e^5 + [-18c(3\phi) + 54c(\phi)]e^4 + [6c(2\phi) \\ &+ 14]e^3 + [10 - 6c(2\phi)]e^9 + [-18c(\phi) + 6c(3\phi)]e^2 \\ &+ [18c(\phi) - 6c(3\phi)]e^8 - 8e + [-22 + 18c(2\phi)]e^7\} \\ &+ \Bigg\{\frac{1}{4}[3c(4\phi) + 9 + 12c(2\phi)]e^9 + \frac{1}{2}[27c(3\phi) + 45c(\phi)]e^2 \\ &+ \frac{1}{2}[39c(\phi) + 3c(5\phi) + 18c(3\phi)]e^8 + [-9 + 3c(2\phi)]e \\ &+ \frac{1}{4}[45 + 72c(2\phi) + 27c(4\phi)]e^7 - 6c(\phi) + \frac{1}{2}[-57c(\phi) \\ &- 6c(5\phi) - 9c(3\phi)]e^6 + \frac{1}{4}[-63c(3\phi) - 168c(2\phi) - 153]e^5 \\ &+ \frac{1}{2}[-15c(\phi) - 36c(3\phi) + 3c(5\phi)]e^4 \\ &+ \frac{1}{4}[33c(4\phi) + 72c(2\phi) + 135]e^3\Bigg\}\Bigg) = 0 \end{aligned} \tag{10}$$

Equation (10) has a few complex solutions and a few real solutions. The real solutions are

$$e = -1 \tag{11}$$

$$e = \pm\frac{\sqrt{6}}{2}, \qquad \phi = \arccos\frac{\sqrt{6}}{3} \tag{12}$$

$$\frac{\partial\phi}{\partial M} = 0 \tag{13}$$

$$e = 0, \qquad \phi = \frac{\pi}{2} \tag{14}$$

Solutions (11) and (12) are ruled out immediately [compare Eq. (9)]. Because of Eq. (8), it is evident that solution (13) is also infeasible ($\partial\phi/\partial M \neq 0\,\forall\, M, e$). Solution (14) is ruled out because there is no physical meaning for steering the system to a constant true anomaly. Thus, $\beta(e, \phi, \partial\phi/\partial M) \neq 0\,\forall\,\boldsymbol{\alpha}_0 \in \mathcal{O} \times \mathbb{S}^4$. Hence, Λ is of full rank $\forall \boldsymbol{\alpha}_0 \in \mathcal{O} \times \mathbb{S}^4$, and $\Sigma_G(\boldsymbol{\alpha}, \boldsymbol{u})$ is accessible $\forall \boldsymbol{\alpha}_0 \in \mathcal{O} \times \mathbb{S}^4$. □

Lemma 1 implies that there exist continuous-time control inputs that steer the vehicle from any initial target orbit to almost all desired target orbits. To implement such a closed-loop control, one can utilize the Jurdjevic–Quinn damping feedback,[10] given by

$$\boldsymbol{u} = -\eta[\nabla V \cdot G]^T, \qquad \eta > 0 \tag{15}$$

where V is a smooth Lyapunov function for the open-loop system $\Sigma_G(\boldsymbol{\alpha}, 0)$. Ilgen has utilized Eq. (15) to derive an orbital transfer controller with limited thrust.[8]

Stabilization Anomalies: The Parabolic Orbital Transfer Problem

In the preceding section we have shown that the accessibility of the GVEs leads to the a low-thrust feedback control law for orbital transfer between two elliptic Keplerian orbits. A natural question is whether the same control law can be used for interplanetary travel by transferring a vehicle from an initial elliptic parking orbit to an Earth-escape parabolic trajectory. To formulate this problem, let the union of all parabolic Keplerian orbits on $\Sigma_G(\boldsymbol{\alpha}, \mathbf{0})$ be parameterized by

$$\Sigma_{\mathrm{GP}} = \{\boldsymbol{\alpha} : a = \infty, e = 1, i \in [0, \pi], \{\Omega, \omega\} \in [0, 2\pi]\} \tag{16}$$

The parabolic continuous-feedback orbital transfer problem is to determine whether there exists a continuous internal asymptotic feedback stabilizer $\boldsymbol{u}(\boldsymbol{\alpha})$ that steers a spacecraft from an initial elliptic Keplerian orbit $\Sigma_0 \in \Sigma_{\mathrm{GE}}$ to a desired parabolic escape trajectory $\Sigma_d \in \Sigma_{\mathrm{GP}}$. Our main result in this context is stated in the following lemma.

Lemma 2: There does not exist a solution to the parabolic continuous-feedback orbital transfer problem.

Proof: Note that in the parabolic case the orbital rate on the parabolic orbit is zero. This means that any set of target orbital elements $\boldsymbol{\alpha}_d \in \Sigma_{\mathrm{GP}}$ will be an equilibrium of $\Sigma_G(\boldsymbol{\alpha}, \mathbf{0})$. Therefore, we can apply Brockett's test, given by theorem 2. First, rewrite Σ_G, given by Eq. (6), using the partition

$$\Sigma_G(\boldsymbol{z}, \boldsymbol{u}) : \begin{cases} \dot{\boldsymbol{z}} = Z\boldsymbol{u} \\ \dot{M} = n + F\boldsymbol{u} \end{cases} \tag{17}$$

$$Z(\boldsymbol{\alpha}_d) = \begin{bmatrix} Z_1(\boldsymbol{\alpha}_d) \\ Z_2(\boldsymbol{\alpha}_d) \end{bmatrix} \tag{18}$$

where $\boldsymbol{z} = [a, e, i, \Omega, \omega]^T$, Z_1 is a 3×3 matrix, and Z_2 is a 2×3 matrix. The necessary condition of theorem 2 is that $\forall \boldsymbol{y} \in B_\delta(\boldsymbol{y}, \mathbf{0}) \exists \boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}, \boldsymbol{x}_d), \exists \boldsymbol{u} \in B_\varepsilon(\boldsymbol{u}, 0)$. Choose the test vector $\boldsymbol{y} = [\mathbf{0}_{1\times 3}, \varepsilon, 0]^T$, where $\varepsilon \neq 0$ is a two-dimensional vector of sufficiently small elements. It is easily verified [e. g., by showing that there does not exist a $\boldsymbol{v} \neq \mathbf{0}$ for which $Z(\boldsymbol{\alpha}_d)\boldsymbol{v} = 0$] that rank $[Z(\boldsymbol{\alpha}_d)] = 3 \forall \boldsymbol{\alpha}_d$, so $Z_1(\boldsymbol{\alpha}_d)\boldsymbol{u} = 0 \Rightarrow \boldsymbol{u} = 0 \Rightarrow Z_2(\boldsymbol{\alpha}_d)\boldsymbol{u} = 0$, which contradicts the assumption $\varepsilon \neq 0$. Thus, there does not exist a solution to the parabolic orbital transfer problem $\forall \boldsymbol{\alpha}_d \in \mathcal{O} \times \mathbb{S}^4$. □

The consequence of lemma 2 is that any continuous, closed-loop controller that utilizes classical osculating orbital elements feedback will fail to steer a spacecraft from an initial elliptic Keplerian orbit to a given parabolic escape trajectory. This interesting observation implies that in the case of the parabolic orbital transfer problem the Gauss variational equations lie in the controllability-stabilizability gap known to exist in nonlinear systems.

Conclusions

This Note presented the problem of low-thrust orbital transfer using continuous orbital elements feedback. The dynamical model used was Gauss's variational equations (GVEs). It was proven that the GVEs render a globally accessible dynamical system.

Several conclusions can be drawn from this research. Most important, using orbital elements as the coordinates for the state-space dynamics results in a regular, globally accessible system, guaranteeing the existence of control inputs that steer the vehicle from any initial target orbit to almost all desired target orbits.

Also, we conclude that closed-loop controllers that utilize classical or equinoctial osculating orbital elements feedback will fail to steer a spacecraft from an initial elliptic Keplerian orbit to a given parabolic escape trajectory. In other words, when the target orbit is parabolic the Gauss variational equations are unstabilizable.

Appendix: Matrix Λ

Denoting $c(\cdot) \triangleq \cos(\cdot)$ and $s(\cdot) \triangleq \sin(\cdot)$, the columns of Λ are

$$[\boldsymbol{g}_0, \boldsymbol{g}_1] = \begin{bmatrix} \dfrac{2a^2ne}{h}\dfrac{\partial\phi}{\partial M}c(\phi) \\ \dfrac{pn}{h}\dfrac{\partial\phi}{\partial M}c(\phi) \\ 0 \\ 0 \\ \dfrac{pn}{he}\dfrac{\partial\phi}{\partial M}s(\phi) \\ \left[-\dfrac{nb}{ahe}\dfrac{\partial\phi}{\partial M}(p + 2re^2) + \dfrac{3\mu e}{na^2h}\right]s(\phi) \end{bmatrix} \tag{A1}$$

$$[\boldsymbol{g}_0, [\boldsymbol{g}_0, \boldsymbol{g}_1]] = \begin{bmatrix} 2\dfrac{a^2n^2e}{h}\left[\dfrac{\partial^2\phi}{\partial M^2}(\phi) - \left(\dfrac{\partial\phi}{\partial M}\right)^2 s(\phi)\right] \\ \dfrac{pn^2}{h}\left[\dfrac{\partial^2\phi}{\partial M^2}c(\phi) - \left(\dfrac{\partial\phi}{\partial M}\right)^2 s(\phi)\right] \\ 0 \\ 0 \\ \dfrac{pn^2}{he}\left[\dfrac{\partial^2\phi}{\partial M^2}s(\phi) + \left(\dfrac{\partial\phi}{\partial M}\right)^2 c(\phi)\right] \\ \dfrac{bn^2}{ahe}\left\{-\left(p + \dfrac{2e^2}{p}\right)\left[c(\phi)\left(\dfrac{\partial\phi}{\partial M}\right)^2 - s(\phi)\dfrac{\partial^2\phi}{\partial M^2}\right] - \dfrac{4e^3r}{p}s^2(\phi)\left(\dfrac{\partial\phi}{\partial M}\right)^2\right\} \\ \quad + \dfrac{3\mu e c(\phi)}{a^2h}\dfrac{\partial\phi}{\partial M}\left(1 + \dfrac{1}{n}\right) \end{bmatrix} \tag{A2}$$

$$
[\boldsymbol{g}_0, \boldsymbol{g}_2] = \begin{bmatrix}
-2\dfrac{na^2e}{h}\dfrac{\partial\phi}{\partial M}s(\phi) \\
\dfrac{ne}{hp}\dfrac{\partial\phi}{\partial M}s(\phi)\left\{r^2[c(\phi)+e] - \dfrac{p}{e}(p+r)\right\} \\
0 \\
0 \\
\dfrac{1}{h}\dfrac{\partial\phi}{\partial M}\left[\dfrac{nr^2}{p}s^2(\phi) + \dfrac{p+r}{e}c(\phi)\right] \\
-\dfrac{b}{ha}\dfrac{\partial\phi}{\partial M}\left[\dfrac{r^2}{p}s^2(\phi) - \dfrac{n}{e}(p+r)c^2(\phi)\right] + \dfrac{2\mu p}{rna^2h}
\end{bmatrix} \tag{A3}
$$

$$
[\boldsymbol{g}_0, [\boldsymbol{g}_0, \boldsymbol{g}_2]] = \begin{bmatrix}
-2\dfrac{a^2en^2}{h}\left[\left(\dfrac{\partial\phi}{\partial M}\right)^2 c(\phi) + s(\phi)\dfrac{\partial^2\phi}{\partial M^2}\right] \\
\dfrac{\mu hr^2e}{a^3p}\Bigg(\left(\dfrac{\partial\phi}{\partial M}\right)^2\left[\dfrac{2re}{p}s^2(\phi)c(\phi) + c^2(\phi) - 2s^2(\phi) + \dfrac{2re^2}{p}s^2(\phi) + es(\phi)\right] \\
\qquad + \dfrac{\partial^2\phi}{\partial M^2}\{s(\phi)[e+c(\phi)]\}\Bigg) \\
0 \\
0 \\
\dfrac{\mu r^2e}{a^3hp}\left\{\left(\dfrac{\partial\phi}{\partial M}\right)^2 s(\phi)\left[\dfrac{2r}{p}s^2(\phi) + \dfrac{3}{e}c(\phi)\right]\right\} + \dfrac{\mu(p+r)}{a^3he}\left[\dfrac{\partial^2\phi}{\partial M^2}c(\phi) - s(\phi)\left(\dfrac{\partial\phi}{\partial M}\right)^2\right] \\
-\dfrac{n^2r^2b}{aph}\left[\dfrac{2re}{p}\left(\dfrac{\partial\phi}{\partial M}\right)^2 s^3(\phi) + 3\left(\dfrac{\partial\phi}{\partial M}\right)^2 s(\phi)c(\phi) + \dfrac{\partial^2\phi}{\partial M^2}s^2(\phi)\right] \\
+\dfrac{bn^2}{ahe}(p+r)\left[\left(\dfrac{\partial\phi}{\partial M}\right)^2 s(\phi) - \dfrac{\partial^2\phi}{\partial M^2}c(\phi)\right] - \dfrac{6\mu e}{a^2h}\dfrac{\partial\phi}{\partial M}s(\phi)
\end{bmatrix} \tag{A4}
$$

$$
[\boldsymbol{g}_0, \boldsymbol{g}_3] = \begin{bmatrix}
0 \\
0 \\
\dfrac{rn}{h}\dfrac{\partial\phi}{\partial M}\left[\dfrac{re}{p}c(\phi+\omega)s(\phi) - s(\phi+\omega)s(\phi)\right] \\
\dfrac{rn}{hs(i)}\dfrac{\partial\phi}{\partial M}[c(\phi+\omega) - s(\phi+\omega)s(\phi)] \\
-\dfrac{rnc(i)}{hs(i)}\dfrac{\partial\phi}{\partial M}\left[\dfrac{re}{p}s(\phi+\omega)s(\phi) + c(\phi+\omega)\right] \\
0
\end{bmatrix} \tag{A5}
$$

$$
[\boldsymbol{g}_0, [\boldsymbol{g}_0, \boldsymbol{g}_3]] = \begin{bmatrix}
0 \\
0 \\
\dfrac{\mu r^2e}{a^2hp}\Bigg\{\left(\dfrac{\partial\phi}{\partial M}\right)^2\left[\dfrac{2re}{p}s^2(\phi) - 2s(\phi+\omega)s(\phi) + c(\phi+\omega)c(\phi) - \dfrac{p}{re}c(\phi+\omega)\right] \\
\qquad + \dfrac{\partial^2\phi}{\partial M^2}\left[c(\phi+\omega)s(\phi) - \dfrac{p}{re}s(\phi+\omega)\right]\Bigg]\Bigg\} \\
\dfrac{\mu r}{a^3hs(i)}\Bigg\{\left(\dfrac{\partial\phi}{\partial M}\right)^2\Bigg[-s(\phi+\omega) + 2ec(\phi+\omega)c(\phi) + 2a^2es(\phi+\omega)s^2(\phi) \\
\qquad + \dfrac{re}{p}s(\phi+\omega)c(\phi)] + \dfrac{\partial^2\phi}{\partial M^2}\left[c(\phi+\omega) + \dfrac{re}{p}s(\phi+\omega)s(\phi)\right]\Bigg]\Bigg\} \\
\dfrac{\mu rc(i)}{a^3hs(i)}\Bigg(\left(\dfrac{\partial\phi}{\partial M}\right)^2\Bigg\{\dfrac{re}{p}\left[-2res(\phi+\omega)s^2(\phi) - 2c(\phi+\omega)s(\phi) - s(\phi+\omega)c(\phi)\right] \\
\qquad + s(\phi+\omega)\Bigg\} - \dfrac{\partial^2\phi}{\partial M^2}\left[c(\phi+\omega) + \dfrac{re}{p}s(\phi+\omega)s(\phi)\right]\Bigg]\Bigg) \\
0
\end{bmatrix} \tag{A6}
$$

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